Goldstein

3.16. Initially, $E_0(r=a) = -\frac{k}{a} + \frac{1}{2}\frac{l^2}{Ma^2} < 0$,

where a is the ~~aphelion~~ perihelion distance, it's characterized by $\dot{r} = 0$. The mass ~~M~~ m hits it tangentially, so $\dot{r}$ remains 0, yet the Energy must be 0 for the orbit to be parabolic, so we demand

$$E'(r=a) = -\frac{k(M+m)}{M\ a} + \frac{1}{2}\frac{(l')^2}{(M+m)a^2} = 0.$$

This expression uniquely determines l', the new angular momentum of the system. The term $\frac{k(M+m)}{M}$ is because by definition of $V(r) = -\frac{k}{r}$, we have adopted the convention that $k = GMm$, thus we need to adjust for the additional mass accordingly via the constant k.

Rewriting the expressions in $\dot{\theta}$ instead of l:

$$E_0(r=a) = -\frac{k}{a} + \frac{1}{2}Ma^2\dot{\theta}^2 < 0,$$

$$E'(r=a) = -\frac{k(M+m)}{Ma} + \frac{1}{2}(M+m)a^2(\dot{\theta}')^2 = 0.$$

Solving for $\dot{\theta}'$: $\frac{k}{Ma} = \frac{1}{2} a^2 (\dot{\theta}')^2$

$$\frac{2k}{Ma^3} = (\dot{\theta}')^2$$

$$\boxed{\sqrt{\frac{2k}{Ma^3}} = \dot{\theta}'}$$

Clearly, our construction has had $\dot{\theta}'$ as the angular velocity of the smaller ~~planet~~ comet with mass m. Letting p' be the linear momentum of the smaller ~~planet~~ comet, completely inelastic collision is characterized by:

$$(M+m) a \dot{\theta}' = Ma\dot{\theta} + p'$$

$$(M+m) a \dot{\theta}' - Ma\dot{\theta} = p'$$

This gives the kinetic energy of the comet:

$$T = \frac{(p')^2}{2m} = \cancel{\left[(M+m)a\dot{\theta}' - Ma\dot{\theta}\right]^2 / 2m}.$$

$$\boxed{\left[(M+m)a\dot{\theta}' - Ma\dot{\theta}\right]^2 / 2m.}$$

this expression is in terms of E_0, which can be turned into eccentricity $e = 1-\alpha$ via

$$e = \sqrt{1 + \frac{2E_0 l^2}{mk^2}} \quad \text{(Goldstein 3.57)}$$

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